Elementary theory of well-structured algebras and nilpotent groups.

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Models and Groups

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Elementary equivalence

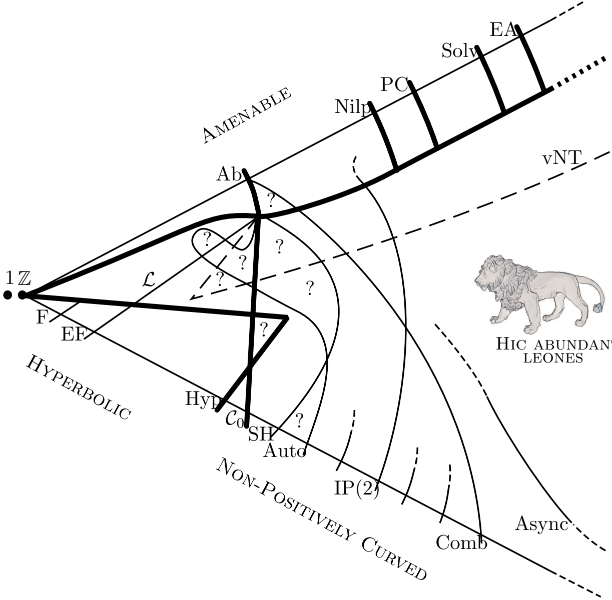
The elementary theory $\mathrm{Th}(G)$ of a group is the set of all sentences which hold in G. Two groups G and H are called elementarily equivalent if $\mathrm{Th}(G) = \mathrm{Th}(H)$.

$$\frac{\text{ALGEBRA}}{\text{ISOMORPHISM}} \leftrightsquigarrow \frac{\text{MODEL THEORY}}{\text{ELEMENTARY EQUIVALENCE}}$$

Problem

Classify groups (in a given class) up to elementary equivalence.

Map of Groups



Abelian Groups

Let A be a torsion-free (without elements of finite order) abelian group

$$\text{Set} \quad \alpha_p(A) = \begin{cases} \dim {}^{A}\!/_{pA}, & \text{if finite;} \\ \infty, & \text{otherwise.} \end{cases}$$

Szmielew characteristic of A is $\chi(A) = (\alpha_2(A), \alpha_3(A), \alpha_5(A), \ldots)$.

Theorem (A, B - torsion free)

$Th(A) = Th(B) \Leftrightarrow \chi(A) = \chi(B)$.

Corollary

1. A - torsion-free, C - divisible, then $Th(A) = Th(A \oplus C)$.
2. All t.f. divisible abelian groups are elementarily equivalent.
3. Two f.g. abelian groups are elementarily equivalent iff they are isomorphic.

Classification of abelian groups up to isomorphism is hopeless.

Elegant classification up to elementary equivalence.

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Nilpotent groups

A group G is called nilpotent if the lower central series:

$$\Gamma_0(G) = G, \quad \Gamma_{i+1}(G) = [\Gamma_i(G), G]$$

is eventually trivial. The minimal i so that $\Gamma_i(G) = 1$ is called the nilpotency class of G.
Equivalently, a group G is nilpotent of class c if

$$G \models \forall x_0, \dots x_c \; [[\dots[x_0, x_1], \dots, x_{c-1}], x_c] = 1.$$

Note that the class of nilpotent groups is a variety.

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Szmielew: Two finitely generated abelian groups are elementarily equivalent iff they are isomorphic.

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Theorem (Zilber, 71)

There are non-isomorphic, but elementarily equivalent finitely generated nilpotent groups of class 2.

Theorem (Oger, 91)

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Central class of nilpotent groups

The study of unitriangular representations of nilpotent groups comes from the classical theory of connected nilpotent Lie groups.

Theorem (Hall, Malcev)

Every finitely generated torsion-free nilpotent group G is a subgroup of some unitriangular matrix group $UT_n(\mathbb{Z})$, $n = n(G)$.

(Note that any finitely generated nilpotent group is a finite extension of a f.g. torsion-free nilpotent group.)

$$UT_n(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a_{11} & a_{12} & \dots & a_{1n-1} \\ 0 & 1 & a_{21} & \dots & a_{2n-2} \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \right\}$$

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Nilpotent groups: role of $UT_3(\mathbb{Z})$

On the one hand $UT_3(\mathbb{Z})$ is unitriangular group, on the other hand it is the free nilpotent group of class 2 and rank 2:

$$1 \to \mathbb{Z} = Z(UT_3(\mathbb{Z})) \to UT_3(\mathbb{Z}) \to \mathbb{Z}^2 \to 1$$

Theorem (Malcev)

Arithmetic is interpretable in $UT_3(\mathbb{Z})$. It follows that the elementary theory of $UT_3(\mathbb{Z})$ is undecidable.

Indeed, the center of $UT_3(\mathbb{Z})$ is $\mathbb{Z} = \{[a,b]^k \mid k \in \mathbb{Z}\}$:

- If $c_1, c_2 \in Z(UT_3(\mathbb{Z}))$, then $c_1 + c_2 = c_1 \cdot c_2$.
- Multiplication in $\mathbb{Z}$: Let $x, y \in Z(UT_3(\mathbb{Z}))$, $x' \in C(a)$, $y' \in C(b)$ be so that $[x', b] = x$ and $[a, y'] = y$, set $x \times y = [x', y']$.
- $0_{\mathbb{Z}}$ is 1 and $1_{\mathbb{Z}}$ is $[a, b]$.

Theorem (Ershov, 72)

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Groups elementarily equivalent to $UT_3(R)$

Let R be a domain, $S \equiv R$. The set $UT_3(R)$ is a nilpotent group (neither torsion-free nor finitely generated). Its centre is R, the ring R is interpretable in $UT_3(R)$.

$$\begin{array}{ccc} 1 \to R \to & UT_3(R) & \to R^2 \to 1 \\ 1 \to S \to & UT_3(S) & \to S^2 \to 1 \\ 1 \to R^* \to & UT_3(R)^* \simeq UT_3(R^*) & \to {R^2}^* \to 1 \end{array}$$

Theorem (Belegradek, 92)

$G \equiv UT_3(R)$ iff $G \simeq UT_3(S, f_1, f_2)$ and $S \equiv R$.

$UT_3(R) = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \right\}$, with the multiplication:

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta').$$

Let $f_1, f_2 : R^+ \times R^+ \to R$ be two symmetric 2-cocycles. New operation on $UT_3(R)$:

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Groups elementarily equivalent to $UT_n(R)$ and to free nilpotent R-groups

$$\begin{array}{cccccccc} 1 \to & R & \to & UT_n(R) & \to & UT_n(R)/R = G & \to 1 \\ 1 \to & Z(G) \simeq R^k & \to & G & \to & G/Z(G) & \to 1 \end{array}$$

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Theorem (Miasnikov-Sohrabi, 2010, 2011)

$G \equiv F_{n,c}(R)$ iff G is an abelian deformation of $F_{n,c}(S)$, $S \equiv R$.

Abelian deformation = deformation of the operation on the group quotient by the commutator by the centre.

Our goal

Find a general approach for both of the above results that can be used in more general settings.

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Nilpotent groups and R-groups

When considering non-commutative groups, it is natural to attempt to extend the idea of a module to the noncommutative case - a group admitting exponents in some ring R.

The chief difficulty lies in attempting to replace the rule $r(x+y) = rx + ry$ (define an action of the ring).

1. If N is a group which is complete and Hausdorff in its p-adic topology, then for any $x \in N$, the homomorphism of $\mathbb{Z}$ into N taking n to x^n extends naturally to a homomorphism of the groups $\mathbb{Z}_p$ of p-adic integers into N. We make N into a group admitting exponents in the ring of p-adic integers.
2. If K is any field of characteristic zero, then an exponent can be defined on $UT_n(K)$:
$$(1+x)^r = 1 + rx + C_r^2 x^2 + \dots$$
3. If K is the field of real numbers, then $UT_n(K)$ is a nilpotent Lie group, and for any $g \in UT_n(K)$, the set of elements of the form g^r defined in this way, is exactly the one-parameter subgroup generated by g.

Nilpotent groups and R-groups

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Nilpotent groups and R-groups

Let R be an associative domain. The ring R gives rise to the category of R-groups. Enrich the language $\mathcal{L}$ with new unary operations $f_r(x)$, one for any $r \in R$. For $g \in G$ and $\alpha \in R$ denote $f_\alpha(g) = g^\alpha$.

Definition

An structure G of the language $\mathcal{L}(R)$ is an R-group if:

- G is a group;
- $g^0 = 1, g^{\alpha+\beta} = g^\alpha g^\beta, g^{\alpha\beta} = (g^\alpha)^\beta$.

As the class of R-groups is a variety, so one has R-subgroups, R-homomorphisms, free R-groups, nilpotent R-groups etc.

Hall R-groups

Definition

Let R be a *binomial* ring. A nilpotent group G of a class m is called a Hall R-group if for all $x, y, x_1, \ldots, x_n \in G$ and any $\lambda, \mu \in R$ one has:

- G is a nilpotent R-group of class m;
- $(y^{-1}xy)^{\lambda} = (y^{-1}xy)^{\lambda}$;
- $x_1^{\lambda} \cdots x_n^{\lambda} = (x_1 \cdots x_n)^{\lambda} \tau_2(x)^{C_2^{\lambda}} \cdots \tau_m(x)^{C_m^{\lambda}}$, where $\tau_i(x) \in \Gamma_{i-1}(F(x))$ is the i-th Petrescu word defined in the free group $F(x)$ by
$$x_1^i \cdots x_n^i = \tau_1(x)^{C_1^i} \tau_2(x)^{C_2^i} \cdots \tau_i(x)^{C_i^i}.$$

Hall R-groups

Proposition (Hall)

Let R be a binomial ring. Then the unitriangular group $UT_n(R)$ and, therefore, all its R-subgroups are Hall R-groups.

Theorem (Merzlyakov 68, Warfield, 76)

A finitely generated, torsion-free R-group is isomorphic to an R-subgroup of $UT_n(R)$, for some positive integer n, if R is a PID, binomial ring.

Groups elementarily equivalent to $UT_3(R)$

Theorem (Belegradek, 92)
$G \equiv UT_3(R)$ *iff* $G \simeq UT_3(S, f_1, f_2)$ *and* $S \equiv R$.

$$\begin{array}{ccc} 1 \to R = Z(UT_3(R)) \to & UT_3(R) & \to R^2 = UT_3(R)/R \to 1 \\ 1 \to S \to & G & \to S^2 \to 1 \\ 1 \to R^* \to & UT_3(R)^* & \to {R^2}^* \to 1 \end{array}$$

It is important that G is not an R-group!

Lie ring/algebra of a nilpotent group

Malcev (1949) proved that there is a category isomorphism between the category of torsion-free nilpotent $\mathbb{Q}$-groups and the category of nilpotent finite-dimensional rational Lie algebras (the isomorphism is given by the Baker-Campbell-Hausdorff formula).

Let G be t.f. nilpotent. Define $Lie(G)$ as follows:

- $Lie(G) = \oplus_{i=1}^{\infty} \Gamma_i / \Gamma_{i+1}$, as an abelian group;
- Let $x = \sum_{i=1}^{\infty} x_i \Gamma_{i+1}$ and $y = \sum_{i=1}^{\infty} y_i \Gamma_{i+1}$, where $x_i, y_i \in \Gamma_i$ are elements of $Lie(G)$. Define a product $\circ$ on $Lie(G)$ by

$$x \circ y = \sum_{k=2}^{\infty} \sum_{i+j=2}^{k} [x_i, y_j] \Gamma_{i+j+1}.$$

Since Γ_i are definable in G, understanding groups $\equiv$ to G is closely related to understanding rings $\equiv$ to $Lie(G)$.
If we are to understand groups $\equiv$ to an R-group G, we should understand rings $\equiv$ to the Lie R-algebra $Lie(G)$.

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Idea of Miasnikov (late 1980's)

1. With an R-algebra A, associate a nice bilinear map $f_A : A/Ann(A) \times A/Ann(A) \to A^2$.
2. A ring $S = S(f_A) = S(R) \supseteq R$, and the S-modules A^2 and $A/Ann(A)$ are interpretable in A in the language of rings.

$$f_A\colon \quad \begin{array}{ccccc} {}^{A}\!/_{Ann(A)} & \times & {}^{A}\!/_{Ann(A)} & \to & A^2 \\ (x + Ann(A) & , & y + Ann(A)) & \mapsto & x \cdot y. \end{array}$$

Model theory of bilinear maps

- Consider a 2-sorted model $U_f = (A/Ann(A), A^2, p_f)$, where p_f is a predicate which describes the map f_A, and $M = A/Ann(A)$, $N = A^2$ are viewed as *abelian groups*.
- Associate to f_A a 3-sorted model $UR_f = (M, N, S, p_f, s_M, s_N)$, where M, N and p_f are as above, $S = S(A)$ is a certain ring containing R, and s_M, s_N are predicates describing the action of S.
- It is clear that U_f is interpretable in UR_f. A theorem of Miasnikov states that (under some conditions on A) *the converse is also true*!

Example

- If $A = R$, then $S = R$.
- If $A = R \times R$, then $S = R \times R$.
- If A is a free nilpotent (Lie or associative) algebra, then $S = R$.
- If A is the direct product of the latter, then $S = R \times R$.

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The ring S

$$U_f = (A/Ann(A), A^2, p_f) \qquad UR_f = (M, N, S, p_f, s_M, s_N)$$

- Let $\mu : R \to P$ be an inclusion of rings. A P-module structure on M is an *enrichment* (wrt μ) if:
  1. Addition is the same in R-module and P-module cases.
  2. And $rm = \mu(r)m$, for every $r \in R$ and $m \in M$.
- An *enrichment* E of f is a pair of enrichments of M and N such that the map f is P-bilinear for the P-modules.
- $E_1 \leq E_2$ if there exists $P_1 \hookrightarrow P_2$ such that the P_2-enrichments of M and N are P_2-enrichments of the P_1-enrichments of M and N.
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Well-structured algebras

An *algebra scheme* $\mathcal{A}(n,\alpha)$, where $n \in \mathbb{N}$, $\alpha = (\alpha_k^{i,j})_{a \le i,j,k \le n} \in \mathbb{Z}^{n^3}$ is a family of algebras that satisfy the following conditions:

- $A(R) \in \mathcal{A}(n,\alpha)$ is an R-algebra, where R is a domain of characteristic 0;
- the underlying module of the algebra $A(R)$ is a free R-module of rank n;
- there is a basis $v_1, \ldots, v_n$ of the module of the algebra $A(R)$ with structural constants α, i.e.

$$v_i v_j = \sum_{k=1,\ldots,n} \alpha_k^{i,j} v_k$$

for all $i, j \in \{1, \ldots, n\}$.

An algebra $A(R) \in \mathcal{A}_{n,\alpha}$ is *well-structured* if:

(WS1) $Ann(A(R)) \subseteq A(R)^2$.

(WS2) $A(R)/A(R)^2$ is a free R-module.

(WS3) $R = P(f_{A(R)}) = S$ is an isomorphism.

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Abelian deformations and characterisation theorem for algebras

$$U_f = (A/Ann(A), A^2, p_f) \qquad UR_f = (M, N, S, p_f, s_M, s_N)$$

Theorem

Let $A(R)$ be a well-structured R-algebra and B be a ring. Then

$$B \equiv A \text{ if and only if } B \simeq QA(T, c)$$

for some ring $T \equiv R$ and some symmetric 2-cocycle $c \in S^2({}^{QA}/_{QA^2}, Ann(QA))$. That is B is an abelian deformation of $A(T)$.

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + c(x_1, y_1)),$$

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Note that if R is a field or a local ring deformations disappear.

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Lie algebras of some groups

Theorem (Belegradek; Miasnikov-Sohrabi; CFKR)

Let R be an integral domain of characteristic zero. And let G be one of the following groups:

- UT_n;
- *free nilpotent group;*
- *directly indecomposable partially commutative nilpotent group.*

Then $Lie_R(G)$ is well-structured.

Characterisation theorem for groups

Theorem (CFKR)

Let G be a Hall R-group so that

- *lower and upper central series of G coincide;*
- *$Lie(G)$ is well-structured.*

Let H be a group, $H \equiv G$. Then H is $QG(S)$ over some ring S such that $S \equiv R$ as rings.

Corollary (Belegradek; Miasnikov-Sohrabi; CFKR)

Let R be a binomial ring. And let G be one of the following groups:

- $UT_n(R)$;
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- free nilpotent R-group;
- directly indecomposable partially commutative nilpotent R-group.

Let H be a group elementarily equivalent to G. Then H is $QG(S)$ over some ring S such that $S \equiv R$ as rings.

THANK YOU!